Two-Parameter Integral Method for Laminar Transpired Thermal Boundary-Layer Flow

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A practical two-parameter polynomial-type integral method is developed for heat transfer associated with laminar transpired boundary-layer flow with transpiration. The method is based on the use of second- and third-order boundary-layer approximations for the distributions in shear stress and heat flux. These approximations are used to establish relationships for the distributions in velocity and temperature and to develop solutions to the integral momentum and energy equations for similar and nonsimilar flows. The accuracy of the method is generally within 3–4%, except near separation where the error can reach 15–20%. The method applies to a wide range of transpiration rates and pressure gradients, including plane and axisymmetric stagnation and separation. In addition, the method provides a fundamental basis for generalization to natural convection and turbulent flow, and a framework for the development of more accurate higher-order multiple-parameter integral methods.

Nomenclature

$a_{n\Delta}$ = coefficients associated with Eq. (18)
B_H = thermal boundary-layer parameter $[\equiv \rho c_P v_o (T_o - T_\infty)/q''_o]$
BP = blowing parameter $[\equiv (v_o/U_\infty)\sqrt{Re_x}]$
b_n = coefficients defined by Eq. (37)
C_n = coefficients associated with Eqs. (6) and (36)
c_n = coefficients defined by Eq. (38)
c_P = specific heat at constant pressure
E_n = coefficients associated with Eq. (27)
F_2 = hydrodynamic boundary-layer parameter defined by Eq. (9)
$F_{2\Delta}$ = thermal boundary-layer parameter defined by Eq. (3b)
f_x = fanning friction factor
H = shape factor $(= \delta_1/\delta_2)$
$J_n(\ )$ = mathematical characteristics defined by Eqs. (B1) and (B4)
$j_{nm}(\ )$ = mathematical characteristics defined by Eq. (30)
k = thermal conductivity
m = hydrodynamic boundary-layer parameter $[\equiv (x/U_\infty)(\mathrm{d}U_\infty/\mathrm{d}x)]$
Nu_Δ = Nusselt number $[q''_o \Delta/k(T_o - T_\infty)]$
n_Δ = thermal boundary-layer parameter $[\equiv x/(T_o - T_\infty)\ (\mathrm{d}T_o/\mathrm{d}x)]$
Pr = Prandtl number
q''_o = wall heat flux
q''_y = heat flux
Re_x = Reynolds number $(\equiv U_\infty x/\nu)$
r = ratio of thermal to hydrodynamic boundary-layer thickness $(\equiv \Delta/\delta)$
r_o = radius of curvature
S = hydrodynamic boundary-layer parameter $(\equiv \tau_o\delta_2/\mu U_\infty)$
S_Δ = thermal boundary-layer parameter $[\equiv q''_o \Delta_2/k(T_o - T_\infty)]$
St = Stanton number
T = temperature distribution
U = dimensionless velocity distribution $(\equiv u/U_\infty)$
U_∞ = freestream velocity
v_o = transpiration rate
Δ_2 = enthalpy thickness defined by Eq. (5)
δ_1 = displacement thickness defined by Eq. (11)
δ_2 = momentum thickness defined by Eq. (12)
Λ = hydrodynamic boundary-layer parameter $[\equiv (\delta^2/\nu)\ (\mathrm{d}U_\infty/\mathrm{d}x)]$
λ = hydrodynamic boundary-layer parameter $[\equiv (\delta_2^2/\nu)\ (\mathrm{d}U_\infty/\mathrm{d}x)]$
λ_Δ = thermal boundary-layer parameter $[\equiv (\Delta_2^2/\nu)\ (\mathrm{d}U/\mathrm{d}x)]$
ξ = dimensionless distance from wall $(\equiv y/\delta)$
ξ_Δ = dimensionless distance from wall $(\equiv y/\Delta)$
ρ = density
Υ = dimensionless termperature distribution $[\equiv (T - T_o)/(T_\infty - T_o)]$
Ω = hydrodynamic boundary-layer parameter $[\equiv -(v_o\delta/\nu)]$
Ω_2 = hydrodynamic boundary-layer parameter $[\equiv (v_o\delta_2/\nu)]$
$\Omega_{2\Delta}$ = thermal boundary-layer parameter $[\equiv -(v_o\Delta_2/\nu)]$

Introduction

INTEGRAL methods for analyzing boundary-layer flow are potentially capable of providing efficient and practical calculations to the point of separation without stepwise solution of the complete boundary-layer equations. Consequently, integral methods continue to provide a useful supplement to numerical finite-difference and finite-element methods. The method of integrals (or weighted-residuals) has received considerable attention over the past few years.[1-16] However, integral methods of this kind generally require the use of a fairly large number of parameters, particularly for near separating flows and turbulent flow. On the other hand, although simple integral methods of the type developed by Pohlhausen,[17] Tani,[18] Truckenbrodt,[19] and Thwaites[20] are in common use, integral methods of this kind have undergone little advancement over the past 20 years or so.

In this connection, several one- and two-parameter integral methods are available in the literature for analyzing nontranspired thermal boundary-layer flow. One of the best-known of these approaches is the two-parameter method by Squire[21] and

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Dienemann.[22] This approach involves the use of polynomial approximations for velocity and temperature of the form

$$U = 2\xi - 2\xi^3 + \xi^4 \tag{1}$$

and

$$\Upsilon = 2\xi_\Delta - 2\xi_\Delta^3 + \xi_\Delta^4 \tag{2}$$

together with the integral energy equation,

$$\frac{1}{r_o}\frac{\mathrm{d}}{\mathrm{d}x}[r_o U_\infty (T_o - T_\infty)\Delta_2] = \frac{q_o''}{\rho c_P} + v_o(T_o - T_\infty) \tag{3a}$$

$$\frac{1}{r_o^2}\frac{U_\infty}{\nu}\frac{\mathrm{d}}{\mathrm{d}x}(r_o\Delta_2)^2 = F_{2\Delta} = 2\left(\frac{S_\Delta}{Pr} - \lambda_\Delta - n_\Delta\frac{Re_{\Delta 2}^2}{Re_x} - \Omega_{2\Delta}\right) \tag{3b}$$

where

$$S_\Delta = \frac{q_o''\Delta_2}{k(T_o - T_\infty)} \tag{4a}$$

$$\lambda_\Delta = \frac{\Delta_2^2}{\nu}\frac{\mathrm{d}U_\infty}{\mathrm{d}_x} \tag{4b}$$

$$n_\Delta = \frac{x}{T_o - T_\infty}\frac{\mathrm{d}T_o}{\mathrm{d}x} \tag{4c}$$

$$\Omega_{2\Delta} = -\frac{v_o\Delta_2}{\nu} \tag{4d}$$

the enthalpy thickness Δ_2 is defined by

$$\Delta_2 = \int_o^\infty U(1-\Upsilon)\,\mathrm{d}y \tag{5}$$

and $v_o = 0$ for no transpiration. This approach also features the use of the Thwaites[20] method for characterizing the momentum transfer. Squire's method provides an accuracy of from 3-10% in the region between stagnation and separation for the case of airflow over a circular cylinder. However, this level of accuracy does not hold for applications involving fluids with low and high values of Prandtl number and cannot be relied upon for other geometries. It should also be noted that the various integral methods presently available in the literature for analyzing thermal boundary-layer flows are only applicable to situations involving uniform wall temperature and do not apply to boundary-layer flow with transpiration. A number of integral methods are discussed in a review paper by Spalding and Pun.[23]

In this connection, a one-parameter polynomial-type integral method has recently been developed for analyzing laminar incompressible boundary-layer flow with transpiration and pressure gradient.[24] In this approach, second- and third-order boundary-layer approximations for the distributions in viscous stress have been used to establish approximations for the velocity distribution for transpired flows of the form

$$U = \sum_{n=0}^{N} C_n\xi^n - C_o e^{-\Omega\xi} \tag{6}$$

where the coefficients C_n are functions of the transpiration parameter Ω,

$$\Omega = -\frac{v_o\delta}{\nu} \tag{7}$$

and the standard pressure gradient parameter Λ,

$$\Lambda = \frac{\delta^2}{\nu}\frac{\mathrm{d}U_\infty}{\mathrm{d}x} \tag{8}$$

This type of approximation has been used to develop solutions to the integral momentum equation,

$$\frac{1}{r_o^2}\frac{U_\infty}{\nu}\frac{\mathrm{d}(r_o\delta_2)^2}{\mathrm{d}x} = F_2 = 2[S - \lambda(2+H) - \Omega_2] \tag{9}$$

where

$$S = \frac{\tau_o\delta_2}{\mu U_\infty} \tag{10a}$$

$$H = \frac{\delta_1}{\delta_2} \tag{10b}$$

$$\lambda = \frac{\delta_2^2}{\nu}\frac{\mathrm{d}U_\infty}{\mathrm{d}x} \tag{10c}$$

$$\Omega_2 = -\frac{v_o\delta_2}{\nu} \tag{10d}$$

and the displacement and momentum thicknesses are given by

$$\delta_1 = \int_o^\infty (1-U)\,\mathrm{d}y \tag{11}$$

$$\delta_2 = \int_o^\infty u(1-U)\,\mathrm{d}y \tag{12}$$

The accuracy of the method is comparable to the accuracy of the one-parameter integral methods of Thwaites[20] and Timman[25] for nontranspired boundary-layer flows. Furthermore, this method provides a basis for generalization to heat transfer and turbulent flow.

The objective of this paper is to develop a practical and reliable polynomial-type integral method for the analysis of transpired thermal boundary-layer flows.

Analysis

Following the pattern established in Ref. 24, a one-parameter integral method is formulated for the energy transfer associated with laminar transpired boundary-layer flow with specified wall temperature or specified wall heat flux by developing boundary-layer approximations for the distributions in molecular conduction heat flux q_y'' and temperature. To accomplish this objective, the distribution in q_y'' is required to satisfy the Couette law

$$\frac{q_y''}{q_o''} = 1 + B_H\Upsilon \tag{13}$$

near the wall, where $B_H = \rho c_P v_o(T_o - T_\infty)/q_o''$, and the conditions

$$\frac{\partial q_y''}{\partial y} = 0 \tag{14}$$

and/or

$$q_y'' = 0 \qquad \text{at } y = \Delta \tag{15}$$

$$\frac{\partial\Upsilon}{\partial y} = 0 \tag{16}$$

and

$$\Upsilon = 1 \qquad \text{at } y = \Delta \tag{17}$$

at the outer edge of the thermal boundary layer. An Nth-order polynomial-type approximation that satisfies these requirements is given by

$$\frac{q_y''}{q_o''} = \sum_{n=0}^{N} a_{n\Delta}\xi_\Delta^n + B_H\Upsilon \tag{18}$$

Table 1 Coefficients $a_{n\Delta}$

N	$a_{o\Delta}$	$a_{1\Delta}$	$a_{2\Delta}$	$a_{3\Delta}$
2	1	0	$-1-B_H$	—
3	1	0	$-3-3B_H$	$2+2B_H$

Table 2 Coefficients $\alpha_{n\Delta}$ and $\gamma_{n\Delta}$

N	$\alpha_{o\Delta}$	$\alpha_{1\Delta}$	$\alpha_{2\Delta}$	$\alpha_{3\Delta}$	$\gamma_{o\Delta}$	$\gamma_{1\Delta}$	$\gamma_{2\Delta}$	$\gamma_{3\Delta}$
2	1	0	−1	—	0	0	1	—
3	1	0	−3	2	0	0	3	−2

with coefficients $a_{n\Delta}$ listed in Table 1 for $N = 2$ and 3. This equation takes the convenient dimensionless form

$$\frac{q''_y \Delta}{k(T_o - T_\infty)} = Nu_\Delta \sum_{n=0}^{N} a_{n\Delta} \xi_\Delta^n - r\Omega Pr \Upsilon \tag{19}$$

or

$$\frac{q''_y \Delta}{k(T_o - T_\infty)} = Nu_\Delta \sum_{n=0}^{\infty} \alpha_{n\Delta} \xi^n + r\Omega Pr \sum_{n=0}^{N} \gamma_{n\Delta} \xi_\Delta^n - r\Omega Pr \Upsilon \tag{20}$$

where $r = \Delta/\delta$, $\Omega = -v_o\delta/\nu$, and

$$a_{n\Delta} = \alpha_{n\Delta} - \gamma_{n\Delta} B_H \tag{21}$$

$$Nu_\Delta = \frac{q''_o \Delta}{k(T_o - T_\infty)} \tag{22}$$

$$Nu_\Delta B_H = -r\Omega Pr \tag{23}$$

The coefficients $\alpha_{n\Delta}$ and $\gamma_{n\Delta}$ are listed in Table 2.

The temperature distribution is expressed in terms of the Fourier's law of conduction,

$$q''_y = -k \frac{\partial T}{\partial y} \tag{24}$$

or

$$\frac{q''_y \Delta}{k(T_o - T_\infty)} = \frac{d\Upsilon}{d\xi_\Delta} \tag{25}$$

Combining Eqs. (19) and (25), the temperature distribution is represented by

$$\frac{d\Upsilon}{d\xi_\Delta} + r\Omega Pr \Upsilon = Nu_\Delta \sum_{n=0}^{N} a_{n\Delta} \xi_\Delta^n \tag{26}$$

The solution to this equation for $\Omega \neq 0$ that satisfies the conditions $\Upsilon = 0$ at $\xi = 0$ and $\Upsilon = 1$ at $\xi = 1$ is of the form (see Appendix A)

$$\Upsilon = \sum_{n=0}^{N} E_n \xi_\Delta^n - E_o e^{-r\Omega Pr \xi_\Delta} \tag{27}$$

where

$$E_n = Nu_\Delta \sum_{m=n}^{N} \alpha_{m\Delta} j_{nm}(APr) \tag{28}$$

$$Nu_\Delta = \frac{N_o + N_1 e^{-r\Omega Pr}}{N_2 + N_3 e^{-r\Omega Pr}} \tag{29}$$

$$j_{nm}(APr) = \frac{(-1)^{m-n} m!}{(APr)^{m-n+1} n!} = \frac{(-1)^{m-n} m!}{(r\Omega Pr)^{m-n+1} n!} \tag{30}$$

with A = $r\Omega$ and

$$N_o = 1 - r\Omega Pr \left[\sum_{n=0}^{N} \sum_{m=n}^{N} \gamma_{m\Delta} j_{nm}(APr) \right]$$

$$N_1 = -r\Omega Pr \sum_{n=0}^{N} j_n(APr)$$

$$N_2 = \sum_{n=0}^{N} \sum_{m=n}^{N} \alpha_m \Delta_{j_{nm}}(APr)$$

$$N_3 = \sum_{n=0}^{N} \alpha_{n\Delta} j_n(APr)$$

With Nu_Δ given by Eq. (29), Eq. (27) expresses the temperature distribution Υ in terms of Λ, Ω, r, and Pr.

The relationships and coefficients associated with the solution for the velocity distribution given by Eq. (6) are given in Tables 3 and 4, respectively. Using these results, expressions are obtained for the enthalpy thickness Δ_2 (see Appendix B).

Setting $\Omega = 0$ for the case of flow over impermeable surfaces, the solution to Eq. (26) is

$$\Upsilon = Nu_\Delta \sum_{n=0}^{N} \frac{a_{n\Delta}}{n+1} \xi_\Delta^{n+1} \tag{31}$$

such that

$$Nu_\Delta = 1 \Big/ \sum_{n=0}^{N} (a_{n\Delta}/n + 1) \tag{32}$$

Combining Eqs. (31) and (32), the polynomial approximation for Υ is put into the form

$$\Upsilon = \sum_{n=1}^{N+1} E_n \xi_\Delta^n \tag{33}$$

where

$$E_n = \frac{a_{n\Delta-1}}{n} Nu_\Delta \tag{34}$$

Table 3 Integral relations associated with Eq. (6) for momentum transfer

$$C = Mo_\delta \sum_{m=n}^{N} a_m j_{nm}(\Omega)$$

$$a_n = \alpha_n - \beta_n \beta_\delta - \gamma_n B_M$$

$$M_{o\delta} = \frac{\tau_o \delta}{\mu U_\infty} = \frac{M_o + M_1 e^{-\Omega}}{M_2 + M_3 e^{-\Omega}}$$

$$M_o = 1 - \sum_{n=0}^{N} \sum_{m=n}^{N} (\beta_m \Lambda + \gamma_m \Omega) j_{nm}(\Omega)$$

$$M_1 = -\sum_{n=0}^{N} (\beta_n \Lambda + \gamma_n \Omega) j_n(\Omega)$$

$$M_2 = \sum_{n=0}^{N} \sum_{m=n}^{N} \alpha_m j_{nm}(\Omega)$$

$$M_3 = \sum_{n=0}^{N} \alpha_n j_n(\Omega)$$

$$\beta_\delta = \frac{\delta}{\tau_o} \frac{dP}{dx} = -\frac{\Lambda}{M_{o\delta}}$$

$$B_M = \frac{\rho v_o U_\infty}{\tau_o} = -\frac{\Omega}{M_{o\delta}}$$

Table 4 Coefficients associated with Eq. (6)

N	a_o	a_1	a_2	a_3	β_o	β_1	β_2	β_3
2	1	β_δ	$-1-\beta_\delta-B_m$	—	0	−1	1	—
3	1	β_δ	$-3-2\beta_\delta-3B_m$	$2+\beta_\delta+2B_m$	0	−1	2	−1

The associated solution for the velocity distribution is given by[24]

$$U = \sum_{n=1}^{N+1} C_n \xi^n \tag{35}$$

where

$$C_n = b_n + \Lambda C_n \tag{36}$$

and

$$b_n = \frac{\alpha_{n-1}}{n} b_1 \tag{37}$$

$$c_n = \frac{\alpha_{n-1}}{n} c_1 + \frac{\beta_{n-1}}{n} \tag{38}$$

With Υ and U represented by Eqs. (33) and (35), expressions are obtained for the enthalpy thickness for nontranspired flow (see Appendix B).

Using the relations developed in this analysis for transpired and nontranspired flows, the parameters Nu_Δ and Δ_2/Δ can be computed as a function of Λ, Ω, r, and Pr by the use of nested do-loops, and the accompanying thermal integral parameters, can be obtained from

$$S_\Delta = Nu_\Delta \frac{\Delta_2}{\Delta} \tag{39a}$$

$$\lambda_\Delta = \left(r \frac{\Delta_2^2}{\Delta} \right) \Lambda \tag{39b}$$

$$\Omega_{2\Delta} = -\frac{v_o \Delta_2}{\nu} = r \frac{\Delta_2}{\Delta} \Omega \tag{39c}$$

$$F_{2\Delta} = 2 \left(\frac{S_\Delta}{Pr} - \lambda_\Delta - \Omega_{2\Delta} - n_\Delta \frac{Re_{\Delta_2}}{Re_x} \right)^2 \tag{39d}$$

When coupled with the corresponding hydrodynamic relations, these results provide a means of establishing effective approximate integral solutions for similar and nonsimilar laminar transpired thermal boundary-layer flows.

Similar Flow

For similar flow the parameters S_Δ, λ_Δ, $\Omega_{2\Delta}$, and $F_{2\Delta}$ are independent of x and the distributions in freestream velocity U_∞ and temperature difference $(T_o - T_\infty)$ are of the form

$$U_\infty = Cx^m = Cx^{\beta/(2-\beta)} \tag{40}$$

$$T_o - T_\infty = Bx^{n_\Delta} \tag{41}$$

for specified wall temperature, where C, B, m, and n_Δ are constants. For these conditions, the integral energy equation for two-dimensional plane flow becomes‡

$$\frac{U_\infty}{\nu} \frac{d\Delta_2^2}{dx} = F_{2\Delta} = \frac{2}{2n_\Delta + 1} \left[\frac{S_\Delta}{Pr} - \lambda_\Delta (1 + n_\Delta) - \Omega_{2\Delta} \right] \tag{42}$$

The solution to this equation for similar conditions is of the form

$$\Delta_2^2 = \frac{F_{2\Delta}}{1-m} \frac{\nu x}{U_\infty} \tag{43}$$

such that

$$\lambda_\Delta = \frac{\Delta_2^2}{\nu} \frac{dU_\infty}{dx} = \frac{m}{1-m} F_{2\Delta} \tag{44}$$

$$St\sqrt{Re_x} = \sqrt{\frac{1-m}{F_{2\Delta}}} \frac{S_\Delta}{Pr} = \frac{S_\Delta/Pr}{\sqrt{\lambda_\Delta + F_{2\Delta}}} \tag{45}$$

‡Equation (42) is obtained from Eq. (3b) by setting

$$n_\Delta \frac{Re_{\Delta_2}^2}{Re_x} = n_\Delta \frac{U_\infty \Delta_2^2}{\nu x} = n_\Delta \frac{F_{2\Delta}}{1-m}$$

and rearranging.

$$r = \frac{\Delta}{\delta} = \frac{\Delta_2}{\Delta_2/\Delta} \frac{\delta_2/\delta}{\delta_2} = \sqrt{\frac{F_{2\Delta}}{F_2}} \frac{\delta_2/\delta}{\Delta_2/\Delta} \tag{46}$$

The accompanying solution to the integral momentum equation is given by[24]

$$\delta_2^2 = \frac{F_2}{1-m} \frac{\nu x}{U_\infty} \tag{47}$$

such that

$$\lambda = \frac{\delta_2^2}{\nu} \frac{dU_\infty}{dx} = \frac{m}{1-m} F_2 \tag{48}$$

$$\Omega_2 = -\frac{v_o \delta_2}{\nu} = -\frac{v_o}{U_\infty} \sqrt{Re_x} \sqrt{\frac{F_2}{1-m}} = -BP \sqrt{\frac{F_2}{1-m}} = -BP\sqrt{\lambda + F_2} \tag{49}$$

and

$$\frac{f_x}{2} \sqrt{Re_x} = \sqrt{\frac{1-m}{F_2}} S = \frac{S}{\sqrt{\lambda + F_2}} \tag{50}$$

where the blowing parameter BP is defined by

$$BP = \frac{v_o}{U_\infty} \sqrt{Re_x} \tag{51}$$

Rearranging Eqs. (48) and (49), m and BP are expressed in terms of the integral parameters by

$$m = \frac{\lambda}{\lambda + F_2} \tag{52}$$

$$BP = -\sqrt{\frac{1-m}{F_2}} \Omega_2 = -\frac{\Omega_2}{\sqrt{\lambda + F_2}} \tag{53}$$

Notice that similar plane stagnation flow is characterized by $m = 1$, $F_2 = 0$, and $F_{2\Delta} = 0$.

For the case of similar axisymmetric stagnation flow for which the radius of curvature r_o is equal to x and U_∞ is given by Eq. (40) with $m = 1$, the solution to Eq. (42) becomes

$$\Delta_2^2 = \frac{F_{2\Delta}}{2} \frac{\nu x}{u_\infty} \tag{54}$$

such that

$$\lambda_\Delta = \frac{F_{2\Delta}}{2} m = \frac{F_{2\Delta}}{2} \tag{55}$$

$$\frac{Nu_x}{\sqrt{Re_x}} = \sqrt{\frac{2}{F_{2\Delta}}} S_\Delta \tag{56}$$

and r is given by Eq. (46). The corresponding solution for momentum transfer is given by[24]

$$\delta_2^2 = \frac{F_2}{2} \frac{\nu x}{U_\infty} \tag{57}$$

$$\lambda = \frac{F_2}{2} \tag{58}$$

$$BP = -\sqrt{\frac{2}{F_2}} \Omega_2 \tag{59}$$

$$\frac{f_x}{2} \sqrt{Re_x} = \sqrt{\frac{2}{F_2}} S \tag{60}$$

With Λ, Ω, n_Δ, and Pr specified, Eqs. (55-60) can be used to calculated m, BP, r, $(f_x/2)\sqrt{Re_x}$, and $Nu_x/\sqrt{Re_x}$ for similar plane flow and similar axisymmetric stagnation flow. Because the thermal integral parameters are functions of Pr and r for transpired flows, the value of r associated with specified values of Pr must be obtained by iteration. To establish the value of r for a given value of Pr, r is first approximated and calculations are made for S_Δ, λ_Δ, Ω, and $F_{2\Delta}$, after which r is recomputed by the use of Eq. (46). Using this approach, successive approximations can be made for r until the desired accuracy is achieved. For nontranspired flows, the thermal integral parameters are independent of Pr, such that an explicit relationship can be obtained for Pr as a function of r and Λ by rearranging Eq. (42) as follows:

$$Pr = \frac{2S_\Delta}{(1+2n_\Delta)F_{2\Delta} + 2\lambda_\Delta(1+n_\Delta)} \tag{61}$$

With Λ, r, and n_Δ specified, the parameters S_Δ, λ_Δ, and $F_{2\Delta}$ can be evaluated and Pr can be computed.

Nonsimilar Flow

The solution of the integral energy and momentum equations for nonsimilar flow generally requires the use of numerical methods. To develop a noniterative numerical finite-difference solution for plane and thin axisymmetric boundary layer, Eq. (3b) is put into the form

$$\frac{1}{r_o^2}\frac{U_\infty}{\nu}\frac{\mathrm{d}}{\mathrm{d}x}\left[r_o\left(\frac{\Delta_2}{\Delta}\right)\Delta\right]^2 = F_{2\Delta} \tag{62}$$

or

$$\frac{\mathrm{d}\Delta^2}{\mathrm{d}x/L} = \left(\frac{\nu}{U_\infty}\right)^2 F_\Delta \tag{63}$$

where

$$F_\Delta = \left(\frac{\Delta}{\Delta_2}\right)^2\left\{F_{2\Delta}\frac{LU_\infty}{\nu} - \left(\frac{U_\infty\Delta}{\nu}\right)^2 \times\left[\frac{\mathrm{d}}{\mathrm{d}x/L}(\Delta_2/\Delta)^2 + 2\left(\frac{\Delta_2}{\Delta}\right)^2\frac{\mathrm{d}r_o/\mathrm{d}x/L}{r_o}\right]\right\} \tag{64}$$

for specified wall temperature. Using the simple Euler forward-difference approximation, the solution to Eq. (64) is written as

$$\Delta_{i+1}^2 = \Delta_i^2 + \left[\left(\frac{\nu}{U_\infty}\right)^2 F_\Delta\right]_i \frac{\Delta x}{L} \tag{65}$$

or

$$Re_{\Delta_{i+1}} = \frac{U_{\infty_{i+1}}}{U_{\infty_i}}\left(Re_{\Delta_i}^2 + F_{\Delta_i} + \frac{\Delta x}{L}\right)^{1/2} \tag{66}$$

where $i = 1,2,3,\ldots$. The corresponding solution to the integral momentum equation for $Re_{\delta_{i+1}}$ is[24]

$$Re_{\delta_{i+1}} = \frac{U_{\infty_{i+1}}}{U_{\infty i}}\left(Re_{\delta_i}^2 + F_{\delta_i}\frac{\Delta x}{L}\right)^{1/2} \tag{67}$$

where

$$F_\delta = \left(\frac{\delta}{\delta_2}\right)^2\left\{F_2\frac{LU_\infty}{\nu} - Re_\delta^2\left[\frac{\mathrm{d}(\delta_2/\delta)^2}{\mathrm{d}x/L} + 2\left(\frac{\delta_2}{\delta}\right)^2\frac{\mathrm{d}r_o/\mathrm{d}x/L}{r_o}\right]\right\} \tag{68}$$

The parameter r is expressed in terms of Re_Δ and Re_δ by

$$r = \frac{\Delta}{\delta} = \frac{Re_\Delta}{Re_\delta} \tag{69}$$

and Λ and Ω are obtained from

$$\Lambda = \frac{Re_\delta^2 m}{Re_x} \tag{70}$$

$$\Omega = -\frac{v_o}{U_\infty}Re_\delta \tag{71}$$

With r, Λ, and Ω known at station $(i+1)$, the integral parameters can be computed, the Stanton number is obtained from

$$St = \frac{Nu_\Delta}{rRe_\delta Pr} \tag{72}$$

and the friction factor from

$$\frac{f_x}{2} = \frac{S}{(\delta_2/\delta)Re_\delta} \tag{73}$$

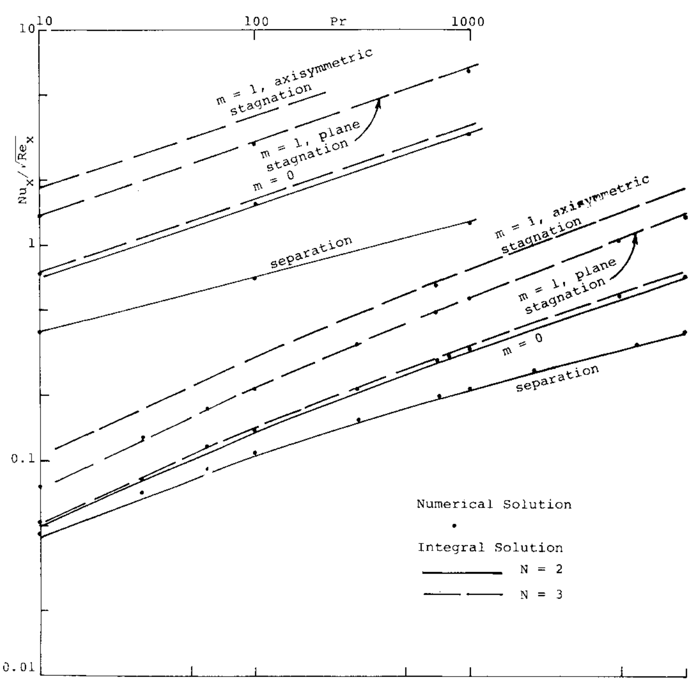

Fig. 1 Calculations for Nusselt number for similar nontranspired boundary-layer flow with uniform wall temperature.

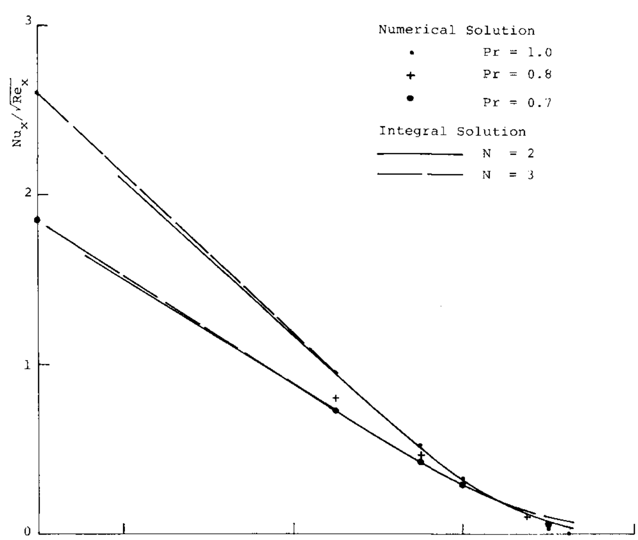

Fig. 2 Calculations for Nusselt number for similar transpired boundary-layer flow with uniform wall temperature and uniform freestream velocity.

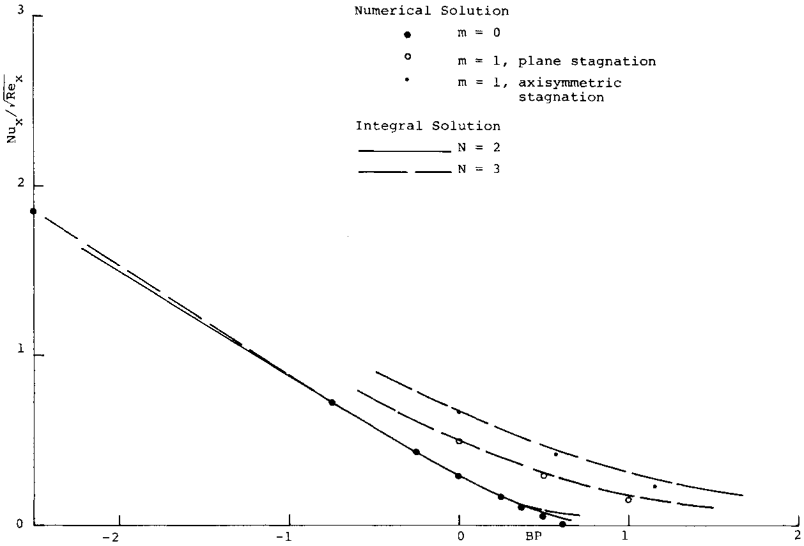

Fig. 3 Calculations for Nusselt number for similar transpired boundary-layer flow with uniform wall temperature and several basic freestream velocity distributions.

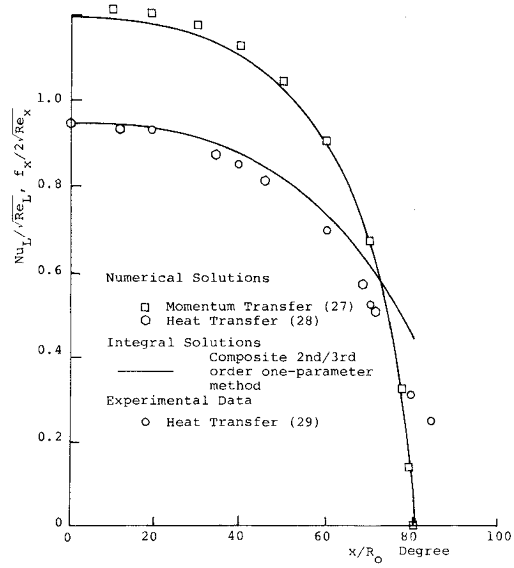

Fig. 4 Friction factor and Nusselt number for flow over a circular cylinder ($Pr = 0.7$.)

Results

Attention is now turned to solution results for heat transfer obtained by means of the second- and third-order integral method developed above for standard Falkner-Skan similar wedge flows with β in the range -0.2 to 2 and for a representative nonsimilar flow.

Integral calculations and exact similarity solutions for Nusselt number Nu_x associated with nontranspired flow are given as a function of Prandtl number Pr in Fig.1 for uniform freestream velocity ($m = 0$), separation ($S = 0$), and plane and axisymmetric stagnation ($m = 1$) conditions. Using the second-order method for retarded flows and the third-order method for accelerating flows, the accuracy is about 3-4% across the entire range of conditions.§ In this connection, the second-order integral method can also be used to achieve this level of accuracy for mild favorable pressure gradient flows ($\Lambda \leq 6$), but breaks down for values of $\Lambda > 6$. On the other hand, use of the standard third-order Pohlhausen integral method for adverse pressure gradients flow near separation results in errors of the order of 20-30%.

Integral and exact similarity solutions for transpired flows with uniform freestream velocity are shown in Fig. 2 for several values of Prandtl number. The accuracy of the calculations ranges from 1-2% for suction to 17% for blowing near separation for the second-order method. The accuracy of the third-order method is comparable, except near separation where the error is about 26%. Notice that the calculations for Nu_x are much less dependent on Pr for blowing than for suction.

To see the effect of pressure gradient on the solution for transpired flows, integral and exact similarity solutions are given in Fig. 3 for plane and axisymmetric stagnation flows and uniform freestream velocity flow with $Pr = 0.7$. The accuracy of the integral calculations for the range of conditions shown is within about 3%.

To demonstrate the capability of the approach for nonsimilar conditions, consideration is given to the classic case of plane flow over a circular cylinder with uniform wall temperature heating, for which the freestream velocity is represented by

$$\frac{U_\infty}{U_o} = 1.814\left(\frac{x}{R_o}\right) - 0.2710\left(\frac{x}{R_o}\right)^3 - 0.04710\left(\frac{x}{R_o}\right)^5 \quad (74)$$

after Hiemenz,[26] where R_o is the cylinder radius and x is the arc length measured from the stagnation point. Because the conditions associated with this flow range from stagnation to separation, the composite approximations are used with $N = 3$ for moderate acceleration and $N = 2$ for mild acceleration and deceleration. Integral and numerical calculations and experimental data for Nusselt number and friction factor are compared in Fig. 4 for nontranspired flow. The accuracy of the integral solutions is within about 3-11% from stagnation to separation.

Conclusions

A polynomial-type integral method has been developed for laminar thermal boundary-layer flow with transpiration and pressure gradient. The approach features the use of second- and third-order approximations for the distributions in heat flux and stress, and involves the solution of the integral momentum and energy equations. The accuracy of the method is generally within about 3%, except in the vicinity of separation where the error can reach 10-20%. Using nested do-loops, this method provides a very efficient and practical means of solving thermal boundary-layer flows for a wide range of conditions, with representative runs on a IBM 3360 requiring about 0.5 s. Furthermore, the approach is quite versatile in that it can be extended to natural convection as well as compressible and turbulent flow. Consequently, this method may prove to be very useful in design work for boundary-layer flows involving transpiration and nonuniform freestream velocity.

Appendix A

Introducing the integration factor $e^{-APr\xi_\Delta}$, the solution to Eq. (26) is written as

$$\Upsilon = Nu_\Delta\left[\sum_{n=0}^{N} a_{n\Delta}\, J_n(APr\,\xi_\Delta)\right] e^{-APr\xi_\Delta} \quad (A1)$$

where $A = r\Omega$,

$$J_n(APr\,\xi_\Delta) = \int_o^{\xi_\Delta} \xi_\Delta^n e^{APr\xi_\Delta}\,\mathrm{d}\xi_\Delta = \frac{\xi_\Delta^n}{APr}\,e^{APr\xi_\Delta} - \frac{n}{APr}\,J_{n-1}\,(APr\,\xi_\Delta) = e^{APr\xi_\Delta}\sum_{m=0}^{n} j_{mn}\,(APr)\,\xi_\Delta^m + j_n\,(APr) \quad (A2)$$

§The accuracy of the calculations for Nu_x is given relative to the Nusselt number for $Pr = 1$ and uniform freestream velocity $Nu_x(o)$; i.e.,

$$\text{Relative error} = \frac{|Nu_x - Nu_{xe}| - |Nu_x(o) - Nu_{xe}(o)|}{Nu_{xe} + Nu_{xe}(o)}$$

where Nu_{xe} represents the exact similarity solution.

and

$$j_{mn}(APr) = \frac{(-1)^{n-m} n!}{(APr)^{n-m+1} m!} \quad (A3)$$

$$j_n(A) = \frac{(-1)^{n+1}}{(APr)} \frac{n!}{n+1} \quad (A4)$$

Combining Eqs. (A1) and (A2) and reordering the double summation, the solution for the temperature distribution becomes

$$\Upsilon = Nu_\Delta \left[\sum_{n=0}^{N} \sum_{m=n}^{N} a_{m\Delta} j_{nm}(APr)\, \xi_\Delta^n + \sum_{n=0}^{N} a_{n\Delta} j_n(APr)\, e^{-r\Omega Pr \xi_\Delta} \right] \quad (A5)$$

where

$$j_{nm}(APr) = \frac{(-1)^{m-n} m!}{(APr)^{m-n+1} n!} = \frac{(-1)^{m-n} m!}{(r\Omega Pr)^{m-n+1} n!} \quad (A6)$$

which leads to Eq. (27).

To obtain an expression for Nu_Δ, Eq. (A5) is written as

$$\Upsilon = Nu_\Delta \left[\sum_{n=0}^{N} \sum_{m=n}^{N} \alpha_{m\Delta} j_{nm}(APr)\, \xi_\Delta^n + \sum_{n=0}^{N} \alpha_{n\Delta} j_n(APr) e^{-r\Omega Pr \xi_\Delta} \right] + r\Omega Pr \left[\sum_{n=0}^{N} \sum_{m=n}^{N} \gamma_{m\Delta} j_{nm}(APr) \xi_\Delta^n + \sum_{n=0}^{N} j_n(APr) e^{-r\Omega Pr \xi_\Delta} \right] \quad (A7)$$

Setting $\Upsilon = 1$ at $\xi_\Delta = 1$ and rearranging, Nu_Δ is given by

$$Nu_\Delta = \frac{1 - r\Omega Pr \left[\sum_{n=0}^{N} \sum_{m=n}^{N} \gamma_{m\Delta} j_{nm}(APr) + \sum_{n=0}^{N} j_n(APr) e^{-r\Omega Pr} \right]}{\sum_{n=0}^{N} \sum_{m=n}^{N} \alpha_{m\Delta} j_{nm}(APr) + \sum_{n=0}^{N} \alpha_{n\Delta} j_n(APr) e^{-r\Omega Pr}} \quad (A8)$$

which is put into the form of Eq. (29).

Appendix B

General Relations for Δ_2/Δ—Transpired Boundary Layers

$$\frac{\Delta_2}{\Delta} = \int_o^1 U(1-\Upsilon)\, d\xi_\Delta = \int_o^1 \left(\sum_{n=0}^{N} C_n r^n \xi_\Delta^n - C_o e^{-r\Omega \xi_\Delta} \right) \times \left(1 - \sum_{n=0}^{N} E_n \xi_\Delta^n + E_o e^{-r\Omega Pr \xi_\Delta} \right) d\xi_\Delta = \sum_{n=0}^{N} \frac{C_n r^n}{n+1} + \frac{C_o}{r\Omega}(e^{-r\Omega} - 1) - \sum_{n=0}^{N} \sum_{m=n}^{N} \frac{C_n E_i r^n}{n+i+1} + \frac{C_o E_o}{r\Omega(Pr+1)} \times [e^{-r\Omega(Pr+1)} - 1] + C_o \sum_{n=0}^{N} E_n J_n(-A) + E_o \sum_{n=0}^{N} C_n r^n J_n(-APr) \quad (B1)$$

for $r \le 1$, where

$$J_o(-A) = \int_o^1 e^{-A\xi_\Delta} d\xi_\Delta = \frac{e^{-A} - 1}{-A} \quad (B2)$$

$$J_n(-A) = \int_o^1 \xi_\Delta^n e^{-A\xi_\Delta} d\xi_\Delta = \frac{e^{-A}}{-A} - \frac{n}{-A} J_{n-1}(-A) \quad (B3)$$

$$\frac{\Delta_2}{\Delta} = \int_o^{1/r} U(1-\Upsilon)\, d\xi_\Delta + \int_{1/r}^1 (1-\Upsilon)\, d\xi_\Delta = \frac{1}{r} \sum_{n=0}^{N} \frac{C_n}{n+1} + \frac{C_o}{r\Omega}(e^{-\Omega} - 1) - \sum_{n=0}^{N} \sum_{m=n}^{N} \frac{C_n E_i}{(n+i+1) r^{i+1}} + \frac{C_o E_o}{r\Omega(Pr+1)} \times [e^{-\Omega(Pr+1)} - 1] + C_o \sum_{n=0}^{N} E_n J_n(-A, r^{-1}) + E_o \sum_{n=0}^{N} \times C_n r^n J_n(-APr, r^{-1}) + 1 - \frac{1}{r} - \sum_{n=0}^{N} \frac{E_n}{n+1} \times \left[1 - \left(\frac{1}{r}\right)^{n+1} \right] - \frac{E_o}{r\Omega Pr}(e^{-r\Omega Pr} - e^{-\Omega Pr}) \quad (B4)$$

for $r \ge 1$, where

$$J_o(-A, r^{-1}) = \int_o^{1/r} e^{-A\xi_\Delta} d\xi_\Delta = \frac{e^{-A/r} - 1}{-A} \quad (B5)$$

$$J_n(-A, r^{-1} = \int_o^{1/r} \xi_\Delta^n e^{-A\xi_\Delta} d\xi_\Delta = \frac{e^{-A/r}}{-Ar^n} - \frac{n}{-A} J_{n-1}(-A, r^{-1}) \quad (B6)$$

Relations for Δ_2/Δ—Nontranspired Boundary Layers

$$\frac{\Delta_2}{\Delta} = \int_o^1 U(1-\Upsilon)\, d\xi_\Delta = \int_o^1 \left(\sum_{n=1}^{N+1} C_n r^n \xi_\Delta^n \right) \left(1 - \sum_{n=1}^{N+1} E_n \xi_\Delta^n \right) \times d\xi_\Delta = \sum_{n=1}^{N+1} \frac{C_n r^n}{n+1} - \sum_{n=1}^{N+1} \sum_{i=1}^{N+1} \frac{C_n E_i r^n}{(n+i+1)} \quad (B7)$$

for $r \le 1$, and

$$\frac{\Delta_2}{\Delta} = \int_o^{1/r} U(1-\Upsilon)\, d\xi_\Delta + \int_{1/r}^1 (1-\Upsilon)\, d\xi_\Delta = \frac{1}{r} \sum_{n=1}^{N+1} \frac{C_n}{n+1} - \sum_{n=1}^{N+1} \sum_{i=1}^{N+1} \frac{C_n E_i}{(n+i+1) r^{i+1}} + 1 - \frac{1}{r} - \sum_{n=1}^{N+1} \frac{E_n}{n+1} \left[1 - \left(\frac{1}{r}\right)^{n+1} \right] \quad (B8)$$

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